> Relaxation and Transport

- till now: focus on thermal equilibrium fluctuation spectrum
  - ⇒ FDT
  - ⇒ collective modes
  - ⇒ kinetic equations (BBGKY)
  - ⇒ Vlasov modes / Landau damping
  - ⇒ test particle model

- now: focus on resistivity and effective Ohm's Law for plasma i.e. J vs E relation

elements:

1. ⇒ collisional resistivity ⇒ how compute collisional transport coefficients?
   - ⇒ transport coefficient as linear response coefficients
   - ⇒ Boltzmann - Landau - Rosenbluth Eqn. via:
     - Lenard-Balescu Eqn. (from TPM)
     - Fokker-Planck Eqn
   - ⇒ Equilibrium
   - ⇒ perturbations ⇒ relaxation
   - ⇒ transport coefficients, incl. σ_{Spitzer}
2. ⇒ beyond collisional resistivity
   - ⇒ why? low collisionality ⇒ current driven ion acoustic instability (CDIA)

⇒ basis of wave instabilities
⇒ wave-particle momentum transfer
⇒ quasilinear theory for mean $\langle f \rangle$ evolution → relaxation, momentum budget
⊕ ⇒ turbulent resistivity

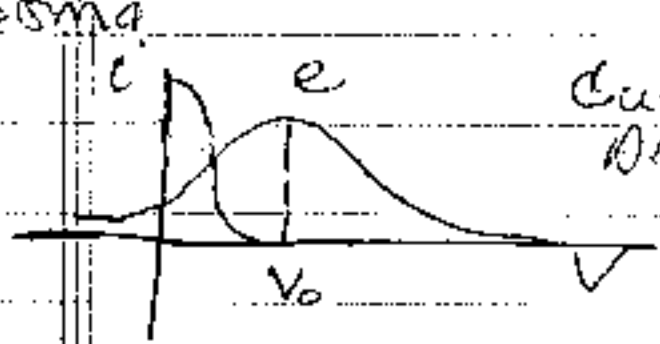
∴ nonlinear $J(E)$ relation

n.b.: Supplementary reading/study of Fokker-Planck theory, Boltzmann Equation, H-theorem is strongly suggested.

(iii) Near-Equilibrium Relaxation : Lenard-Balescu Equation

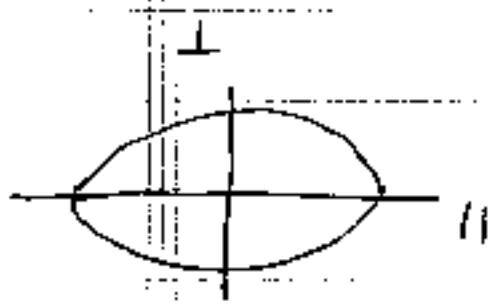
→ here, seek understand evolution of $\langle f \rangle$ (i.e. transport, relaxation) in near-equilibrium plasma

i.e.



Current Driven Ion Acoustic → resistivity

$V_0 < V_{0crit}$



$T_{\parallel} > T_{\perp}$ → isotropization/thermalization

→ systematic procedure for evolution equation of $\langle f \rangle$ → i.e. beyond Landau Collision Integral ?
or where from Landau Collision Integral.

Now,

velocity space flux → drives relaxation

$$\frac{\partial \langle f \rangle}{\partial t} = -\frac{q}{m}\frac{\partial}{\partial v}\langle E\, \delta f \rangle = -\frac{\partial}{\partial v}\Gamma_v$$

i.e. "transport"

$q = -|e|$ ⟺ electrons, FYI

→ how does mean $\langle f \rangle$ evolve in response to small inhomogeneities.

$$\delta f = f^c + \tilde{f}$$

↳ discreteness fluctuation (every particle both a (pea / element of pea soup))

Q.L. response (coherent)

simple mean field theory with total δf

$\therefore$

$$\frac{\partial \langle f \rangle}{\partial t} = -\frac{\partial}{\partial v}\left\langle \frac{q}{m}\tilde{E} f^c \right\rangle - \frac{\partial}{\partial v}\left\langle \frac{q}{m}\tilde{E}\tilde{f} \right\rangle$$

$$= \frac{-\frac{q}{m}\tilde{E}_{k,\omega}\frac{\partial \langle f \rangle}{\partial v}}{-i(\omega - kv)}$$

so

$$\Gamma_v = \underbrace{-D(v)\frac{\partial \langle f \rangle}{\partial v}}_{\text{(quasi-linear) diffusion with T.P.M. spectrum}} + \underbrace{F\langle f \rangle}_{\text{drag due discreteness}}$$

(i.e. random kicks from thermal fluctuations)

(i.e. ala' wake, wave emission exerts force on emitter)

$$D = \sum_{k,\omega} \frac{q^2}{m^2} k^2 \langle \tilde{\phi}^2 \rangle_{k,\omega}\, \pi\delta(\omega - kv)$$

{i.e. note ω-summation

$$F = -\frac{q}{m}\sum_{k,\omega} ik \frac{\langle \tilde{\phi}\tilde{f} \rangle_{k,\omega}}{\langle f \rangle}$$

test particle correlator

$$\langle \tilde{\phi}^2 \rangle_{k,\omega} = \left(\frac{4\pi n_0 q}{k^2}\right)^2 \frac{\langle \tilde{n}^2 \rangle_{k,\omega}}{|\epsilon(k,\omega)|^2}$$

Coulomb spectrum

screening

$$\langle \tilde{n}^2 \rangle_{k,\omega} = \int d\underline{v} \; \frac{2\pi}{n} \delta(\omega - \underline{k}\cdot\underline{v}) \langle f(\underline{v}) \rangle \qquad (3D)$$

$$= \int dv \; \frac{2\pi}{n} \delta(\omega - kv) \langle f(v) \rangle \qquad (1D)$$

so

dim-less

$$\boxed{\langle \tilde{\phi}^2 \rangle_{k,\omega} = \left(\frac{4\pi n_0 q}{k^2}\right)^2 \left(2\pi / |k| v_{Te} n_0\right) \frac{\langle \bar{f}(\omega/k v_{Te}) \rangle}{|\epsilon(k,\omega)|^2}}$$

$\hookrightarrow$ determines diffusion D

$$\langle \tilde{\phi} \tilde{f} \rangle_{k,\omega} = \left(\frac{4\pi n_0 q}{k^2}\right) \int dv \; \frac{\langle \tilde{f}\tilde{f} \rangle_{k,\omega}}{\epsilon^*(k,\omega)}$$

$$= \left(\frac{4\pi n_0 q}{k^2}\right) \frac{1}{\epsilon^*(k,\omega)} \int dv \; \frac{\delta(\omega - kv)}{n_0} \langle f \rangle \delta(v_1 - v_2)$$

$$= \frac{4\pi n_0 q}{k^2} \frac{1}{\epsilon^*(k,\omega)} \, 2\pi \delta(\omega - kv) \frac{\langle f \rangle}{n_0}$$

$\hookrightarrow$ determines drag F

→ velocity space current

so $$\frac{\partial \langle f\rangle}{\partial t} = -\frac{\partial}{\partial v} J(v)$$

$$J(v) = -D\frac{\partial \langle f\rangle}{\partial v} + F\langle f\rangle$$

$$= \sum_{k,\omega}\left(\frac{4\pi n_0 q}{k^2}\right)\frac{q}{m}\left(\frac{2\pi\,\delta(\omega-kv)}{n_0|\epsilon(k,\omega)|^2}\right) * k\left\{\left(\frac{4\pi n_0 q}{k^2}\right)\frac{\pi k}{|k|v_{te}}\left(\frac{q}{m}\right)\right.$$

$$\left.\langle \bar{f}(\omega/k)\rangle\frac{\partial\langle f\rangle}{\partial v} + \epsilon_{IM}(k,\omega)\langle f\rangle\right\}$$

but

$$\epsilon_{IM} = -\frac{\pi\omega_p^2}{|k|^2}\frac{k}{|k|v_{te}}\left.\frac{\partial\langle f\rangle}{\partial v}\right|_{\omega/k} + \epsilon_{IM}^{ion}$$

so, in 1D:

$$J(v) = \sum_{k,\omega}\left(\frac{\omega_p^2}{k^2}\right)^2\left(\frac{2\pi^2 k}{n_0 k^2 v_t}\right)\frac{\delta(\omega-kv)}{|\epsilon(k,\omega)|^2}\left\{\langle\bar{f}(\omega/kv_t)\rangle\frac{\partial\langle f\rangle}{\partial v}\right.$$

$$\left. - \langle f(v)\rangle\left.\frac{\partial\langle\bar{f}(\omega/kv_t)\rangle}{\partial v}\right|_{\omega/k} + \epsilon_{IM}^{ion}\langle f(v)\rangle\right\}$$

$$= D_{e,e}\frac{\partial\langle f_e\rangle}{\partial v} + F_{e,e}\langle f_e(v)\rangle + F_{e,i}\langle f_e(v)\rangle$$

$D_{e,e}$ → diffusion of electrons (by electron-discreteness-generated fluctuations)

$F_{e,e}$ → electron-electron drag (on electrons by discreteness-generated fluctuations)

$F_{e,i}$ → electron-ion drag via ion damping of collective modes

of course: $\frac{\partial \langle f \rangle}{\partial t} = -\frac{\partial}{\partial v} J(v)$ $\qquad u = \omega/k$

$$J(v) = \sum_{k,\omega} \left(\frac{\omega_p^2}{k^2 v}\right)^2 \left(\frac{2\pi^2 k}{n_0 k^2 v_T}\right) \frac{\delta(v-\omega/k)}{|\epsilon(k,\omega)|^2} \left\{ \langle \bar{f}(u) \rangle \frac{\partial \langle f \rangle}{\partial v} - \left.\frac{\partial \langle \bar{f} \rangle}{\partial v}\right|_u \langle f(v) \rangle + \epsilon_{IM}^{ion}(k,\omega) \langle f(v) \rangle \right\}$$

Now, observe:

→ in 1D

$$D_{e,e} + F_{e,e} = \delta(v-u)\left(\langle \bar{f}(u) \rangle \frac{\partial \langle f \rangle}{\partial v} - \langle f(v) \rangle \left.\frac{\partial \langle f \rangle}{\partial v}\right|_u \right)$$

$$= 0 \; !$$

i.e. → electron – electron diffusion and electron – electron drag cancel

→ sole survivor is electron – ion drag
electrons relax only by coupling to ions

Physics: → 1D collision, conserving $\left\{\begin{array}{l}\text{energy}\\ \text{momentum}\end{array}\right.$, must have final state = initial state
∴ no relaxation

→ interchange v, u, obtain $\partial_t \langle f \rangle = -\partial_t \langle f \rangle$,
so $\partial_t \langle f \rangle = 0$

so, in 1D, for stable plasma:

$$\boxed{J_e(v) = \sum_{k,\omega} \left(\frac{\omega_p^2}{k^2}\right)^2 \left(\frac{2\pi k}{n_0 k v_t}\right) \frac{\delta(v-\omega/k)}{|\epsilon(k,\omega)|^2}\, \epsilon_{IM}^{\text{ion}} \langle f_e(v) \rangle}$$

→ in 3D $\qquad B_0 = 0$

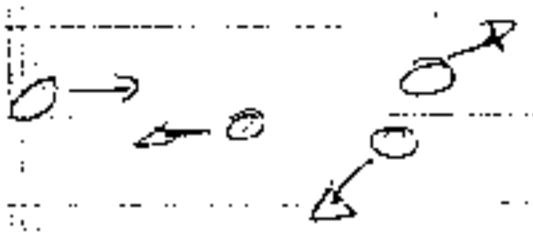
$$\underline{J}(v) = \sum_{k,\omega} \left(\frac{4\pi n_0 q}{k^2}\right)^2 \frac{q}{m} \frac{2\pi\,\delta(\omega - \underline{k}\cdot\underline{v})}{n_0 |\epsilon(k,\omega)|^2} \;*$$

$$\underline{k} \left\{ \left(\frac{4\pi n_0 q}{k^2}\right) \left(\frac{\pi}{4\epsilon_0}\right) \frac{q}{m} \int d\underline{v}'\, \delta(\omega - \underline{k}\cdot\underline{v}') \langle f(\underline{v}')\rangle\, \underline{k}\cdot \frac{\partial \langle f\rangle}{\partial \underline{v}} \right.$$

$$\left. - \left(\frac{4\pi n_0 q}{k^2}\right) \frac{\pi}{4\epsilon_0} \frac{q}{m} \int d\underline{v}'\, \delta(\omega - k\cdot\underline{v}')\, k\cdot \frac{\partial\langle f\rangle}{\partial \underline{v}'} \langle f(\underline{v})\rangle \right\}$$

∴ → in 3D can satisfy $\omega = \underline{k}\cdot\underline{v}$ while changing the direction of $\underline{v}$

Coll.



electron - electron relaxation can occur.

⟺ in 3D, can keep $\omega = k\cdot v$ while changing direction of $\underline{v}$.

iv.) Relation of Test Particle Model / Lenard-Balescu Theory, to Landau Collision Integral

Recall Landau Collision Integral: (will address later) → generic F-P Eqn

$$C(f) = -\frac{\partial}{\partial \underline{p}} \underline{J}_\alpha(\underline{p})$$

transition prob., glancing collision

$$\underline{J}_\alpha(\underline{p}) = \sum_{spc.} \int_{q_\alpha > 0} d^3 q \int w(\underline{p}, \underline{p}'; \underline{q}) \left[f(\underline{p}) \frac{\partial f'(\underline{p}')}{\partial p'_\beta} - f'(\underline{p}') \frac{\partial f(\underline{p})}{\partial p_\beta} \right] q_\alpha q_\beta \, d^3 p'$$

$$= \sum_{species} \int \left[f(\underline{p}) \frac{\partial f'}{\partial p'_\beta}(\underline{p}') - f'(\underline{p}') \frac{\partial}{\partial p_\beta} f(\underline{p}) \right] B_{\alpha\beta} \, d^3 p'$$

$$B_{\alpha\beta} = \frac{1}{2} \int d\sigma \, q_\alpha q_\beta \, |\underline{v} - \underline{v}'|$$

and for Coulombic interaction:

i.e. $\dfrac{\partial \langle f \rangle}{\partial t} = C(f)$

$$= \frac{\partial}{\partial \underline{p}} \left(\underline{F} \langle f(\underline{p}) \rangle \right) + \frac{\partial}{\partial \underline{p}} \cdot \underline{\underline{D}} \cdot \frac{\partial \langle f \rangle}{\partial \underline{p}}$$

$$\sigma = \frac{4\pi (ee')^2}{\mu^2 (\underline{v}-\underline{v}')^4} L \quad , \quad L = \int dx/x$$

Coulomb Logarithm → cut-off

$$B_{\alpha\beta} = \frac{2\pi (ee')^2}{|\underline{v}-\underline{v}'|} L \left[\delta_{\alpha\beta} - \frac{(v_\alpha - v_\alpha')(v_\beta - v_\beta')}{(\underline{v}-\underline{v}')^2} \right]$$

Key elements:

→ glancing collisions / weak deflections

→ molecular chaos, i.e. $f(1,2) = f(1)\, f(2)$

So, natural to compare:

	Landau Collision Theory	Test Particle Model, Lenard-Balescu Theory
"Scenario"	(test) particle scattered by equilibrium distribution of field particles.	test particle scattered by screened ballistic spectrum $\tilde{V} \sim \omega/k$, due other particles.
Correlation	uncorrelated test, field particles via molecular chaos $\langle f(1,2)\rangle = \langle f(1)\rangle\langle f(2)\rangle$	discrete uncorrelated test particles $\langle \tilde{f}\tilde{f}\rangle = \frac{\langle f\rangle}{n}\delta(x_-)\,\delta(v_-)$
Collective effects	no screening $\Rightarrow$ must cut-off Coulomb logarithm "by hand"	screening via f^c and $1/\lvert\epsilon(k,\omega)\rvert^2 \Rightarrow$ screening length appears naturally
nonlinearity	weak deflection $\lvert\underline{q}\rvert \ll \lvert\underline{p}\rvert$	unperturbed orbits

suggests that incorporation of screening into Landau collision integral should recover L-B. theory.

→ Recall Landau Collision integral:

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial p_\alpha} J_\alpha$$

detailed balance small momentum transfer $|q|$ ⇒

$$J_\alpha = \sum_{spc.} \int B_{\alpha\beta} \Big[\underbrace{f(p) \frac{\partial f'(p')}{\partial p_\beta'}}_{\text{drag}} - \underbrace{f'(p') \frac{\partial f(p)}{\partial p_\beta}}_{\text{diffusion}} \Big] d^3p'$$

p' ≡ field particle
p ≡ test particle } momenta

$$B_{\alpha,\beta} \equiv \frac{1}{2} \int d\sigma \, q_\alpha q_\beta |\underline{v}-\underline{v}'|$$

$$= \frac{2\pi (ee')^2}{|\underline{v}-\underline{v}'|} L \Big[\delta_{\alpha\beta} - \frac{(v_\alpha - v_\alpha')(v_\beta - v_\beta')}{(v-v')^2} \Big]$$

transverse form

$\underline{q} \perp \underline{v}$

as $d\sigma = 2\pi b \, db$

$$\frac{ee'}{b} \sim \frac{\mu v_{rel}^2}{2} \Rightarrow b^2 \sim 4 \left[\frac{(ee')}{\mu v_{rel}^2} \right]^2$$

$\sigma \sim \pi b^2$

$l_{mfp} \sim 1/n\sigma$

$\nu = v_{th}/l_{mfp}$

$$d\sigma = \frac{4(ee')^2 d\rho}{\mu^2 \; V_{rel}^4 \, x^4}$$

$$= \frac{4(ee')^2 dx}{\mu^2 V_{rel}^4 x^3}$$

$$\therefore \quad d\sigma_{Trans} = x^2 d\sigma$$

To connect Landau Collision theory and Lenard-Balescu Theory, calculate $B_{\alpha\beta}$ in test particle model and recover screened Landau result.

So

→ calculate $\hat{\phi}$ due to screened test particle of velocity $\underline{v}'$

→ calculate deflection of particle of velocity $\underline{v}$ due $\hat{\phi}$.

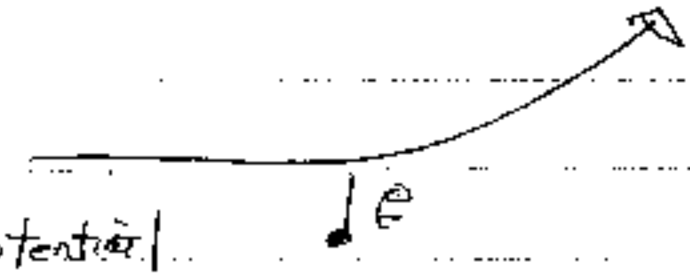
Now $-\nabla \cdot \underline{\underline{\epsilon}} \, \underline{\nabla} \phi = 4\pi e' \, \delta(\underline{x} - \underline{v}'t)$

$$\hat{\phi}_{k,\omega} = \frac{4\pi e'}{k^2 \epsilon(k,\omega)} \, 2\pi \, \delta(\omega - \underline{k}\cdot\underline{v}')$$

$$\phi_{k}(t) = \int \frac{d\omega}{(2\pi)} \frac{4\pi e'}{k^2 \epsilon(k,\omega)} \, 2\pi \, \delta(\omega - \underline{k}\cdot\underline{v}') \, e^{-i\omega t}$$

$$= \frac{4\pi e'}{k^2 \epsilon(k, \underline{k}\cdot\underline{v}')} \, e^{-i\underline{k}\cdot\underline{v}' t}$$

<u>For deflection</u> :



interaction-potential

$$\underline{q} = \int_{u.p.o.} dt \left(- \frac{\partial U}{\partial \underline{r}} \right) = - \int \frac{\partial U}{\partial r}$$

$\underline{r} = \rho + \underline{v} t$

↳ impact param.

$$U = e\phi$$

$$= 4\pi e e' \int_{u.p.o} \frac{d^3k \, e^{i\underline{k}\cdot\underline{r}} \, e^{-i\underline{k}\cdot\underline{v}' t}}{k^2 \epsilon(\underline{k}, \underline{k}\cdot\underline{v}')}$$

$$= 4\pi e e' \int \frac{d^3k \, e^{i\underline{k}\cdot\rho} \, e^{i\underline{k}\cdot(\underline{v}-\underline{v}')t}}{k^2 [\epsilon(\underline{k}, \underline{k}\cdot\underline{v}')]}$$

⊗

$$\underline{q} = 4\pi e e' \int \frac{d^3k}{(2\pi)^3} \frac{-i\underline{k} \, e^{i\underline{k}\cdot\rho}}{k^2 \epsilon(\underline{k}, \underline{k}\cdot\underline{v}')} \, 2\pi \, \delta\left(\underline{k}\cdot(\underline{v}-\underline{v}')\right)$$

from:

$$\int dt \, e^{i\underline{k}\cdot(\underline{v}-\underline{v}')t}$$

using $\delta(\underline{k}\cdot(\underline{v}-\underline{v}')) = \delta(k_\parallel |\underline{v}-\underline{v}'|)$

$$= \frac{1}{|\underline{v}-\underline{v}'|}\,\delta(k_\parallel)$$

$\therefore$

$$\boxed{q = 4\pi e e' \int \frac{d^2k_\perp}{(2\pi)^2}\, \frac{-i\underline{k}_\perp\, e^{i\underline{k}_\perp\cdot\underline{\rho}}}{k^2\,\epsilon(\underline{k},\underline{k}\cdot\underline{v}')\,|\underline{v}-\underline{v}'|}}$$

momentum transfer

for $B_{\alpha,\beta}$

$$B_{\alpha\beta} = \int d\nabla_f\, \frac{q_\alpha q_\beta}{2}\,|\underline{v}-\underline{v}'|$$

$$= \int d^2\rho\; q_\alpha q_\beta\, |v-\underline{v}'|$$

$$\int d^2\rho\, q_\alpha q_\beta \sim \int d^2\rho\; e^{i\underline{k}_\perp\cdot\underline{\rho}}\, e^{i\underline{k}_\perp'\cdot\underline{\rho}}$$

$$\sim (2\pi)^2\,\delta(\underline{k}_\perp + \underline{k}_\perp')$$

$$\int d^2k_\perp'\, \delta(\underline{k}_\perp + \underline{k}_\perp') = 1$$

so

$$B_{\alpha,\beta} = 2e^2e'^2 \int d^2k_\perp \frac{\underline{k}_{\perp\alpha}\, \underline{k}_{\perp\beta}}{|k_\perp^2\, \epsilon(\underline{k}_\perp, \underline{k}_\perp\cdot\underline{v}')|^2\, |\underline{v}-\underline{v}'|}$$

$\Rightarrow$

$$\boxed{B_{\alpha\beta} = 2e^2e'^2 \int d^2k_\perp \frac{k_{\perp\alpha}\, k_{\perp\beta}}{|k_\perp^2\,\epsilon(\underline{k}_\perp, \underline{k}_\perp\cdot\underline{v}')|^2\, |\underline{v}-\underline{v}'|}}$$

<u>Note:</u>

i.) $\epsilon(\underline{k}, \underline{k}\cdot\underline{v}')$ $\rightarrow$ dynamic screening factor
$\rightarrow$ evaluates ϵ induced by (ballistically) propagating source

ii.) note if $\epsilon \to 1$ (no collective screening)

$$B \sim \int d^2k \frac{k_\perp^2}{|\epsilon|^2 k_\perp^4} \sim \int dk_\perp\, k_\perp \frac{k_\perp^2}{k_\perp^4} \sim \int \frac{dk_\perp}{k_\perp}$$

$$\sim \ln(k_{\perp max}/k_{\perp min})$$

$\rightarrow$ recovers Coulomb logarithm.

if $k, \omega \to 0$

$$\epsilon = 1 + 1/k^2\lambda_D^2$$

$k_\perp^2\epsilon \sim k_\perp^2 + 1/\lambda_D^2$ $\rightarrow$ <u>no</u> long range divergence.

iii) limits of integration:

$k_{min} \sim 1/\lambda_D$ (via ϵ)

$k_{max} \sim \frac{1}{2} \mu v_{rel}^2 / ee'$ (distance of closest approach)

Now, can re-write $B_{\alpha,\beta}$ as

$$B_{\alpha,\beta} = 2(ee')^2 \int_{-\infty}^{+\infty} d\omega \int_{k<k_{max}} d^3k \; \delta(\omega - \underline{k}\cdot\underline{v})\, \delta(\omega - \underline{k}\cdot\underline{v}') \, \frac{k_\alpha k_\beta}{k^4 |\epsilon(k,\omega)|^2}$$

recovers L-B, theory noting:

one $\delta(\omega - k \cdot v')$ ⇒ propagator in $\langle \tilde{f}\tilde{f} \rangle_{k,\omega}$

2nd $\delta(\omega - \underline{k}\cdot\underline{v})$ ⇒ propagator in Q.L. terms

(b) Applications – Dynamic Screening – { Collective Enhancement of Collisional Relaxation }

Consider form $B_{\alpha\beta}$:

$$B_{\alpha\beta} = 2(q_\alpha q_\beta)^2 \int_{-\infty}^{+\infty}\int_{k<k_{max}} \delta(\omega - \underline{k}\cdot\underline{v})\,\delta(\omega - \underline{k}\cdot\underline{v}')\,\frac{k_\alpha k_\beta\, d^3k\, d\omega}{k^4 |\epsilon(\underline{k},\omega)|^2}$$

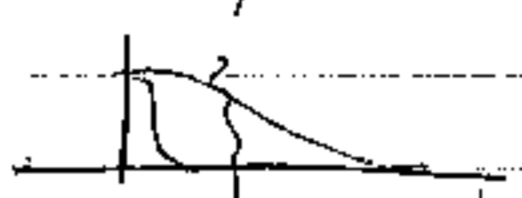
i.e. $\underline{k}\cdot\underline{v} = \underline{k}\cdot\underline{v}' \Rightarrow \underline{k}\cdot(\underline{v}-\underline{v}') = 0$

Consider stable, 2-species plasma. Then have two collective resonances (i.e. weakly damped waves) (no shift):

(1) $\rightarrow$ electron plasma waves; $\omega/k > V_{Te}$

(2) $\rightarrow$ ion acoustic waves; $V_{Ti} < \frac{\omega}{k} < V_{Te}$
(no shift f_e)

(1) $\rightarrow$ tail of $\langle f \rangle_e$ $\Rightarrow$ relatively few particles, little role in collision dynamics

(2) $\rightarrow$ if $T_e \gg T_i$  not 

weak damping
$\langle f \rangle$ small

heavy ion, electron damping

$\therefore$ for $T_e \gg T_i$, ion acoustic resonance may enhance collisional relaxation (weakly damped modes)

To show, exploit 'pole approximation':
(collective resonance enhancement of B)

$$\frac{1}{|\epsilon|^2} = \frac{1}{|\epsilon_r|^2 + |\epsilon_{IM}|^2}$$

$$\cong \frac{1}{\left[\left((\omega-\omega_k)\left.\frac{\partial\epsilon}{\partial\omega}\right|_{\omega_k}\right)^2 + |\epsilon_{IM}|^2\right]}$$

(damping → resonance linewidth)

$$\cong \frac{\pi}{|\epsilon_{IM}|}\,\delta(\epsilon_r)$$

$\delta(\epsilon_r)$ → wave resonance

i.e. $\epsilon_r = 0$ → resonance location

ϵ_{IM} → resonance size/width

and note for electron - electron collisions:

$\omega \ll \underline{k}\cdot\underline{v}$, $\underline{k}\cdot\underline{v}'$; due $\omega < \underline{k}\cdot\underline{v}_{Te}$ ordering

⇒ so

$$B_{\alpha\beta} \cong 2\pi q^4 \int_{-\infty}^{+\infty}\!\!\int \delta(\underline{k}\cdot\underline{v})\,\delta(\underline{k}\cdot\underline{v}')\,\delta(\epsilon_r)\,\frac{k_\alpha k_\beta}{|\epsilon_{IM}|}\,d^3k\,d\omega$$

Change variables:

$\kappa = \underline{k}\cdot\hat{n}$ (kappa) $\hat{n}$ unit along $\underline{v}\times\underline{v}'$

$k_1 = \underline{k}\cdot\underline{v}$

$k_2 = \underline{k}\cdot\underline{v}'$

then: $d^3k = d\kappa\, dk_1\, dk_2 / |\underline{v}\times\underline{v}'|$

so

$\Rightarrow$

$$B_{\alpha\beta} = \frac{2\pi e^4 n_\alpha n_\beta}{|\underline{v}\times\underline{v}'|}\; 2\int_{k>0} dk \int_{-\infty}^{+\infty} d\omega\, \frac{\delta(\epsilon_r(k,\omega))}{k^3\,|\epsilon_{IM}|}$$

i.e. collapse k_1, k_2 integrals

Now,
$$\left.\begin{aligned} \epsilon_r &= 1 - \frac{\omega_{pi}^2}{\omega^2} + \frac{1}{k^2\lambda_D^2} \\ \omega &= kc_s \Big/ (1+k^2\lambda_{De}^2)^{1/2} \end{aligned}\right\}\ \text{ion-acoustic wave}$$

$$\Rightarrow\quad \epsilon_{IM} = \sqrt{\frac{\pi}{2}}\,\underbrace{\frac{\omega}{k^3}}_{\substack{\uparrow}}\Bigg(\overbrace{\frac{1}{\lambda_{De}^2 v_{Te}}}^{\substack{\text{electron}\\ \text{L.D.},\ \omega\ll k v_{Te}}} + \overbrace{\frac{1}{\lambda_{Di}^2 v_{Ti}}\, e^{-\omega^2/2k^2 v_{Ti}^2}}^{\text{I.L.D.}}\Bigg)$$

dominant contribution from shortwavelengths $(k\lambda_{De}>1)$
$\rightarrow$ (i.e. max $1/|\epsilon_{IM}|$)

$\therefore$
$$\begin{cases} \omega \cong \omega_{pi} \\ \epsilon_{IM} = \sqrt{\dfrac{\pi}{2}}\,\dfrac{\omega_{pi}}{k^3}\left(\dfrac{1}{\lambda_{De}^2 v_{Te}} + \dfrac{1}{\lambda_{Di}^2 v_{Ti}}\, e^{-1/k^2\lambda_{Di}^2}\right) \end{cases}$$

$\Rightarrow$

$$\delta(\epsilon_r) = \delta\left(1-\omega_{pi}^2/\omega^2\right) \cong \frac{1}{2}\,\omega_{pi}\left[\delta(\omega-\omega_{pi})+\delta(\omega+\omega_{pi})\right]$$

$$\Rightarrow \boxed{B_{\alpha\beta} = \frac{4\pi e^4 \omega_{pi}\, n_\alpha n_\beta}{|\underline{v}\times\underline{v}'|} \int \frac{dk}{k^2 |\epsilon_{I,f}(\omega_{pi};k)|}}$$

$$\varepsilon = k^2 \lambda_{Di}^2$$

$$\epsilon_{IM} = \sqrt{\frac{\pi}{2}}\, \frac{\omega}{k^3} \left\{ \frac{1}{\lambda_{De}^2 v_{Te}} + \frac{1}{\lambda_{Di}^2 v_{Ti}} e^{-\omega_{pi}^2/2k^2 v_{Ti}^2} \right\}$$

$$\therefore \left\{ \begin{aligned} B_{\alpha\beta} &= n_\alpha n_\beta \frac{2\sqrt{2\pi}\, e^4 v_{Te} \lambda_{De}^2}{|\underline{v}\times\underline{v}'|\ \lambda_{Di}^2} \int d\varepsilon \Big/ \left[1 + \exp\left(-\frac{1}{2\varepsilon} + \frac{L_1}{2}\right) \right] \\ L_1 &= \ln\left((T_e/T_i)(v_{Te}^2/v_{Ti}^2) \right) \end{aligned} \right.$$

Now:

(i) $v_{Ti} < \frac{\omega}{k} < v_{Te} \Rightarrow \overset{(\omega_{pi}\lambda_{Di}/\omega_{pe}\lambda_{De})^2}{} < \varepsilon < 1$

(ii) $L_1 \gg 1 \Rightarrow$ expand $O(1/L_1)$

i.e. dominant contribution when:

$$\exp\left(-1/2\varepsilon + \frac{L_1}{2}\right) \ll 1 \Rightarrow \varepsilon \lesssim 1/L_1$$

note: $\left[1 + \exp(-1/2\varepsilon + L_1/2)\right]^{-1} \equiv$ i.e. denominator new:

$$\boxed{B_{\alpha\beta} = n_\alpha n_\beta \left[\frac{2\sqrt{2\pi}\, e^4 v_{Te}\, \lambda_{De}^2}{|v \times v'|\, \lambda_{Di}^2}\right] (1/L)}$$

Now;

$$B_{\alpha\beta} = \underbrace{B_{\alpha\beta}^{\text{collective}}}_{\substack{\text{above } (v \ll v_{Te}) \\ \text{(collective resonance dominant)}}} + \underbrace{B_{\alpha\beta}^{\text{coulomb}}}_{\substack{(v \sim v_{Te}) \\ \text{(ballistic spectrum dominant)}}}$$

as peaks in spectrum not coincident

$$B_{\alpha\beta}^{\text{coulomb}} = \frac{2\pi e^4}{|v - v'|} L \left[\delta_{\alpha\beta} - \frac{(v_\alpha - v_\alpha')(v_\beta - v_\beta')}{|v - v'|^2}\right]$$

$$\cong \frac{2\pi e^4}{v_{Te}} L_{\text{coulomb}}$$

$$B^{\text{collective}} \geq B^{\text{coulomb}} \quad \text{if:}$$

$$\boxed{\frac{T_e}{T_i L_1} \geq \underbrace{L_{\text{coulomb}}}_{\text{Coulomb log.}}}$$

Criterion for dominance of collective effect enhanced scattering

(need $T_e \gg T_i$)